

# Counting Dyons in $\mathcal{N} = 8$ String Theory

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## Abstract

A recently discovered relation between 4D and 5D black holes is used to derive exact (weighted) BPS black hole degeneracies for 4D  $\mathcal{N} = 8$  string theory from the exactly known 5D degeneracies. A direct 4D microscopic derivation in terms of weighted 4D D-brane bound state degeneracies is sketched and found to agree.

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## 1. Introduction

In this paper, we deduce an exact formula for the modified elliptic genus of string theory in four dimensions with  $\mathcal{N} = 8$  supersymmetry. The modified elliptic genus, as we review below, provides a weighted count of BPS states of  $\mathcal{N} = 8$  string theory. We derive a formula for it using a recently proposed exact relation [1] between 4D and 5D BPS degeneracies, together with the known degeneracies [2] in 5D. In addition we sketch a direct microscopic counting of D0-D2-D4 bound states which gives the same result. Our hope is that this example will provide a useful laboratory for testing the string theory relations recently proposed in e.g.[3].

Some years ago an explicit formula for the elliptic genus for BPS states in 4D  $\mathcal{N} = 4$  theories was presciently conjectured [4]. This formula was recently derived using the 4D-5D connection in [5]. The present work is an extension of [5] to 4D  $\mathcal{N} = 8$  theories. Previous work in this direction includes [6,7,8].

In the next section we review the 5D index defined and computed in [2]. In section 3 we use the 4D-5D connection to derive the 4D index. In section 4 we sketch how this expression should follow (for one element of the U-duality class of black holes) from a microscopic analysis.

## 2. Review of the 5D modified elliptic genus

In this section, we want to summarize the work of reference [2] on counting the microstates of  $1/8$  BPS black holes in five dimensions. These can be realized in string theory as the usual D1-D5-momentum system of type IIB on  $T^4 \times S^1$ , with  $Q_1$  D1-branes,  $Q_5$  D5-branes and integral  $S^1$  momentum  $n$ . The reason that microstate counting of this system is more difficult than for  $K3$  compactification is because the usual supersymmetric index that counts these microstates, the orbifold elliptic genus of  $Hilb^k(K3)$  with  $k = Q_1 Q_5$ , vanishes when  $K3$  is replaced with  $T^4$ . In [2], this difficulty was overcome by defining

(and then computing) a new supersymmetric index  $\mathcal{E}_2$ , closely related with the elliptic genus, which is nonvanishing for  $T^4$ . We will refer to this new supersymmetric index as the modified elliptic genus of  $Hilb^k(T^4)$ . It is defined to be

$$\mathcal{E}_2^{(k)} = Tr \left[ (-1)^{2J_L^3 - 2J_R^3} 2(J_R^3)^2 q^{L_0} \bar{q}^{\bar{L}_0} y^{2J_L^3} \right] \quad (2.1)$$

where the trace is over states of the sigma model with target space  $Hilb^k(T^4)$ .<sup>1</sup> Here  $J_L^3$  and  $J_R^3$  are the left and right half-integral  $U(1)$  charges of the CFT, and they are identified with generators of  $SO(4)$  rotations of the transverse  $R^4$ . The  $S^1$  momentum is  $n = L_0 - \bar{L}_0$ . The usual elliptic genus is given by the same formula but without the  $2(J_R^3)^2$  factor; it is these two insertions of  $J_R^3$  that make  $\mathcal{E}_2$  nonvanishing for  $T^4$ .

As for  $K3$ , here it is convenient to define a generating function for the modified elliptic genus:

$$\mathcal{E}_2 = \sum_{k \geq 1} p^k \mathcal{E}_2^{(k)} \quad (2.2)$$

In [2], this was shown to be given by the following sum

$$\mathcal{E}_2(p, q, y) = \sum_{s, k, n, \ell} s(p^k q^n y^\ell)^s \widehat{c}(nk, \ell) \quad (2.3)$$

with the sum running over  $s, k \geq 1, n \geq 0, \ell \in \mathbb{Z}$ . Note that the  $\bar{q}$  dependence has dropped out – only the  $\bar{L}_0 = 0$  states contribute to the modified elliptic genus. Of course, the index must have this property in order to count BPS states, since the BPS condition is equivalent to requiring  $\bar{L}_0 = 0$ .

It was furthermore shown in [2] that the integers  $\widehat{c}(nm, \ell)$  are the coefficients in the following Fourier expansion

$$Z(q, y) \equiv -\eta(q)^{-6} \vartheta_1(y|q)^2 = \sum_{n, \ell} \widehat{c}(n, \ell) q^n y^\ell \quad (2.4)$$

where  $\eta(q)$  is the usual Dedekind eta function, and  $\vartheta_1(y|q)$  is defined by the product formula

$$\vartheta_1(y|q) = i(y^{1/2} - y^{-1/2}) q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^n) \quad (2.5)$$

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<sup>1</sup> A free sigma model on  $R^4 \times T^4$  is factored out here, and our definition differs by a factor of 2 from [2].

Finally, it was observed in [2] that  $\widehat{c}(n, \ell)$  actually only depends on a single combination of parameters  $4n - \ell^2$ :

$$\widehat{c}(n, \ell) = \widehat{c}(4n - \ell^2) \quad (2.6)$$

Using (2.6) in (2.3) yields

$$\mathcal{E}_2(p, q, y) = \sum_{s, k, n, \ell} s(p^k q^n y^\ell)^s \widehat{c}(4nk - \ell^2) \quad (2.7)$$

When  $(k, n, \ell)$  are coprime,  $\widehat{c}(4nk - \ell^2)$  counts BPS black holes with  $k = Q_1 Q_5$ ,  $S^1$  momentum  $n$  and spin  $J_L^3 = \frac{\ell}{2}$ , multiplied by an overall  $(-)^{\ell}$  and summed over  $J_R^3$  weighted by  $2(J_R^3)^2(-)^{2J_R^3}$ :

$$\widehat{c}(4nk - \ell^2) \Big|_{(k, n, \ell) \text{ coprime}} = (-)^{\ell} \sum_{J_R, \text{BPS states}} 2(J_R^3)^2(-)^{2J_R^3} \quad (2.8)$$

When they are not coprime, the black hole can fragment, and the situation is more complicated due to multiple contributions in  $\mathcal{E}_2$  [2]. In this paper we will always avoid this complication by choosing coprime charges.

We should note that  $Z(q, y)$  is also the modified elliptic genus of  $T^4$ , i.e.

$$\mathcal{E}_2^{(1)} = \sum_{n, \ell} \widehat{c}(n, \ell) q^n y^\ell = Z(q, y). \quad (2.9)$$

This corresponds to the coprime D1-D5 system with  $k = 1 = Q_1 = Q_5$ . By writing

$$Z(q, y) = \sum_m \widehat{c}(4m) q^m \sum_k q^{k^2} y^{2k} + \sum_m \widehat{c}(4m - 1) q^m \sum_k q^{k^2 + k} y^{2k+1} \quad (2.10)$$

and using (2.4) along with the standard Fourier expansion of the theta function

$$\vartheta_1(y|q) = i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \quad (2.11)$$

one can reorganize the generating functions for  $\widehat{c}$  as

$$\begin{aligned} \sum_m \widehat{c}(4m) q^m &= -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 + m}, \\ \sum_m \widehat{c}(4m - 1) q^m &= q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2}. \end{aligned} \quad (2.12)$$

These expressions will be analyzed microscopically below in section 4.

### 3. The 4D modified elliptic genus

In this section we use the conjecture of [1] to transform the 5D degeneracies into 4D ones. The fact that the  $\widehat{c}$  coefficients depend only on the combination  $4nk - \ell^2$  is very encouraging, for the following reason. We expect the 1/8 BPS 5D degeneracies to be related to degeneracies of 1/8 BPS black holes in 4D, and in 4D U-duality implies [9] that the black hole entropy must depend on the unique quartic invariant of  $E_{7,7}$ , the so-called Cremmer-Julia invariant [10]. In an  $\mathcal{N} = 4$  language, this invariant takes the form

$$\mathcal{J} = q_e^2 q_m^2 - (q_e \cdot q_m)^2 \quad (3.1)$$

where  $q_e$  and  $q_m$  are the electric and magnetic charge vectors for  $\mathcal{N} = 4$  BPS states. (See e.g. [5] for details on the notation.) This is precisely the dependence of  $\widehat{c}$  on  $n$ ,  $m$ ,  $\ell$ , provided we identify

$$k = \frac{1}{2}q_e^2, \quad n = \frac{1}{2}q_m^2, \quad \ell = q_e \cdot q_m. \quad (3.2)$$

Note that from the purely 5D point of view, there was no obvious reason that  $\widehat{c}$  should depend only on the combination  $4nk - \ell^2$  as there is no 5D U-duality which mixes spins with charges.

Let us now derive the identification (3.2) from the dictionary of [1], beginning from the IIB spinning 5D D1-D5- $n$  black hole of the previous section. First we T-dual on  $S^1$  to obtain a black hole with spin  $\frac{\ell}{2}$ , F-string winding  $n$ ,  $Q_1$  D0-branes, and  $Q_5$  D4-branes. Now T-dual so that there are  $Q_1 + Q_5$  D2 branes with intersection number  $Q_1 Q_5 = k$  on the  $T^4$ . Next we compactify on a single center Taub-NUT, whose asymptotic circle we identify as the new M-theory circle. The result is three orthogonal sets of  $(n, Q_1, Q_5)$  D2-branes on  $T^6$ ,  $\ell$  D0-branes, and one D6-brane. For IIA D-brane configurations with D0, D2, D4, D6 charges  $(q_0, q_{ij}, p^{ij}, p^0)$ , where  $i = 1, \dots, 6$  runs over the  $T^6$  cycle and  $p^{ij} = -p^{ji}$ ,  $q_{ij} = -q_{ji}$   $\mathcal{J}$  reduces to<sup>2</sup>

$$\begin{aligned} \mathcal{J} = & \frac{1}{12} (q_0 \epsilon_{ijklmn} p^{ij} p^{kl} p^{mn} + p^0 \epsilon^{ijklmn} q_{ij} q_{kl} q_{mn}) \\ & - p^{ij} q_{jk} p^{kl} q_{li} + \frac{1}{4} p^{ij} q_{ij} p^{kl} q_{kl} - (p^0 q_0)^2 + \frac{1}{2} p^0 q_0 p^{ij} q_{ij}. \end{aligned} \quad (3.3)$$

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<sup>2</sup> See e.g. [11], equation (66), and take  $p^0 = p_{87}$ ,  $p_{8i} = 0$ , etc. Our definition of  $\mathcal{J}$  differs from that of [11] by a sign.

For our D0-D2-D6 configuration, we can pick a basis of cycles without loss of generality such that the nonzero charges are

$$p^0 = 0, \quad q_0 = \ell, \quad q_{12} = -q_{21} = n, \quad q_{34} = -q_{43} = Q_1, \quad q_{56} = -q_{65} = Q_5 \quad (3.4)$$

Then (3.3) reduces to

$$\mathcal{J} = 4nk - \ell^2, \quad (3.5)$$

which, as stated above, is exactly the argument of (2.3).

According to [1] the weighted degeneracy of the 4D black hole resulting from U-duality and Taub-NUT compactification equals that of the original 5D black hole, when  $J_R^3$  in (2.8) is identified with the generator  $J^3$  of  $\mathbb{R}^3$  rotations in 4D. Note that, since  $\mathcal{J}$  is odd if and only if  $\ell$  is, we may trade  $(-)^{\ell}$  for  $(-)^{\mathcal{J}}$  in (2.8). Therefore, for fixed coprime charges, the weighted 4D BPS degeneracy depends only on the the Cremmer-Julia invariant and is given by

$$\sum_{J^3, BPS \text{ states}} 2(J^3)^2 (-)^{2J^3} = (-)^{\mathcal{J}} \widehat{c}(\mathcal{J}). \quad (3.6)$$

Note that, although this formula for the 4D BPS degeneracy was derived assuming a specific D6-D2-D0 configuration, it applies to all D-brane configurations by U-duality.

As a first check on this conjecture, we note that for large charges  $\widehat{c}(\mathcal{J}) \sim e^{\pi\sqrt{\mathcal{J}}}$ . From the supergravity solutions  $\text{Area} = 4\pi\sqrt{\mathcal{J}}$ , so there is agreement with the Bekenstein-Hawking entropy.

As an example, let's consider the modified elliptic genus for the D4-D0 black hole on  $T^6$ , in which we fix the D4 charges and sum over D0 charge  $q_0$ . Consider the  $T^6$  of the form  $T^2 \times T^2 \times T^2$  with  $\alpha_1, \alpha_2, \alpha_3$  being the three 2-cycles associated with the  $T^2$ 's. Let  $A^1, A^2, A^3$  be the dual 4-cycles. We shall consider the D4-brane wrapped on the cycle  $[P] = A^1 + A^2 + A^3$ . Its triple self-intersection number is  $D = P \cdot P \cdot P = 6$ . From (3.3) we have

$$\mathcal{J} = 4q_0. \quad (3.7)$$

We then have

$$\mathcal{E}_2(q) = \sum_{q_0 \in \mathbb{Z}} \widehat{c}(4q_0) q^{q_0} = -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 + m}. \quad (3.8)$$

according to (2.12).

A straightforward generalization of this example is the D4-D2-D0 system, where we wrap  $(q_1, q_2, q_3)$  D2 branes on the 2-cycles  $(\alpha_1, \alpha_2, \alpha_3)$ . In this case, the Cremmer-Julia invariant becomes

$$\mathcal{J} = 4(q_0 + q_1 q_2 + q_1 q_3 + q_2 q_3) - (q_1 + q_2 + q_3)^2 \quad (3.9)$$

and the sum over  $q_0$  produces

$$\mathcal{E}_2(q) = \sum_{q_0 \in \mathbb{Z}} (-1)^{\mathcal{J}} \widehat{c}(\mathcal{J}) q^{q_0} = \begin{cases} -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 + m - \frac{1}{4} \tilde{\mathcal{J}}} & q_1 + q_2 + q_3 \text{ even} \\ -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 - \frac{1}{4} \tilde{\mathcal{J}} - \frac{1}{4}} & q_1 + q_2 + q_3 \text{ odd} \end{cases} \quad (3.10)$$

where  $\tilde{\mathcal{J}} = 4(q_1 q_2 + q_1 q_3 + q_2 q_3) - (q_1 + q_2 + q_3)^2$ . Now let us turn to the 4D derivation of (3.8) and (3.10).

#### 4. Microscopic derivation in 4D

In this section we sketch a derivation of (3.8) and (3.10) using a 4D microscopic analysis. The derivation is not complete because, as we will discuss below, we ignore some potential subtleties associated to the fact that  $P$  is not simply connected. In principle it should be possible to close this gap. A microscopic description of  $T^6$  black holes using the M-theory picture of wrapped fivebranes has been given in [8], adapting the description given in [12] for a general Calabi-Yau, in terms of a  $(0, 4)$  2D CFT living on the M-theory circle. For uniformity and simplicity of presentation, we here will use the IIA description in which fivebrane momenta around the M-theory circle become bound states of D0 branes to D4 branes.

As above (3.7) we examine the special case of the D4-D0 system wrapped on  $[P] = A^1 + A^2 + A^3$ . The D4-D0 system can be described in terms of the quantum mechanics of  $q_0$  D0-branes living on the D4-brane world volume  $P$ . The D4-brane world volume  $P$  is holomorphically embedded in the  $T^6$ . One can compute its Euler character,  $\chi(P) = 6$ . It follows from the Riemann-Roch formula that the only modulus of  $P$  is the overall translation in  $T^6$ .<sup>3</sup> Since  $\chi(P) = 6$ ,  $P$  has  $4 + 2b_1$  2-cycles. By the Lefschetz hyperplane theorem we have  $b_1(P) = b_1(T^6) = 6$ , and therefore  $b_2(P) = 16$ . All but one of the 2-cycles

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<sup>3</sup> The dual line bundle  $\mathcal{L}_P$  of the divisor  $P$  has only one holomorphic section. However as  $T^6$  is not simply connected, the line bundle  $\mathcal{L}_P$  is not only determined by  $c_1(\mathcal{L}_P) = [P]$ . In fact the translation of  $T^6$  takes it to a different line bundle.

come from the intersection of  $P$  with  $\binom{6}{4} = 15$  4-cycles in  $T^6$ . We will be mostly interested in 3 of these, denoted by  $\tilde{\alpha}_i$ , corresponding to intersections of  $A^i$  with  $P$ . Turning on fluxes along these three 2-cycles corresponds to having charges of D2-branes wrapped on the  $\alpha_i$ 's. Their intersection numbers are

$$\tilde{\alpha}_i \cdot \tilde{\alpha}_j = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases} \quad (4.1)$$

There is, however, one extra 2-cycle in  $P$ , which we shall denote by  $\beta$ , that does not correspond to any cycle in the  $T^6$ .

One can show from the adjunction formula that  $c_1(P)$  is Poincaré dual to  $-(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3)$ . It then follows from Hirzebruch signature theorem that

$$\sigma(P) = -\frac{2}{3}\chi(P) + \frac{1}{3}\int_P c_1^2 = -2. \quad (4.2)$$

We conclude that the intersection form on  $P$  is odd (and that  $P$  is not a spin manifold). Essentially the unique way to extend (4.1) to an odd rank 4 unimodular quadratic form is to have an extra 2-cycle  $\gamma$  with

$$\gamma \cdot \tilde{\alpha}_i = 1, \quad \gamma \cdot \gamma = 1. \quad (4.3)$$

Now if we choose  $\beta = 2\gamma - \sum \tilde{\alpha}_i$ , we have

$$\beta \cdot \tilde{\alpha}_i = 0, \quad \beta \cdot \beta = -2. \quad (4.4)$$

Note that  $(\tilde{\alpha}_i, \beta)$  is not an integral basis for  $H_2(P, \mathbb{Z})$ , yet  $\beta$  is the smallest 2-cycle that doesn't intersect  $\tilde{\alpha}_i$ . The total intersection form on  $P$  is the sum of this rank 4 form together with 6 copies of  $\sigma_1$  coming from the 12 other 2-cycles in  $P$ .

Now one can turn on gauge field flux on the D4-brane world volume along  $\beta$ , which does not correspond to any D2-brane charge. This flux nevertheless induces D0-brane charge. There is a subtlety in the quantization of this flux. As well known, the curvature of the D4-brane world volume induces an anomalous D0-brane charge  $-\chi(P)/24 = -\frac{1}{4}$ . In order that the total D0 charge be integral the flux along the cycle  $\beta$  on the D4-brane must be half-integer, i.e. of the form  $(m + \frac{1}{2})\beta$ . The total induced D0-brane charge is  $\Delta q_0 = -\frac{1}{2}(m + \frac{1}{2})^2 \beta \cdot \beta - \frac{1}{4} = m^2 + m$ , which is indeed an integer.<sup>4</sup>

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<sup>4</sup> In the M-theory picture the anomalous D0 charge is the left-moving zero point energy  $-\frac{cL}{24} = -\frac{1}{4}$ , and the 2-cycle fluxes correspond to momentum zero modes of scalars on a Narain lattice.



We ignore here the facts arising from nonzero  $b_1(P)$  that there is a moduli space of flat connections as well as overall  $T^6$  translations which must be quantized. These factors are treated in the language of the 2D CFT in [8]. They are found to lead to extra degrees of freedom which are however eliminated by extra gauge constraints. A complete microscopic derivation, not given here, would have to show that a careful accounting of these factors give a trivial correction to our result.

It is now straightforward to reproduce (3.8). Each D0-D4 bound state is in a hypermultiplet which contributes minus one to  $Tr[2(J^3)^2(-)^{2J^3}]$ . Counting the number of ways of distributing  $n$  D0-branes among the  $\chi(P) = 6$  ground states of the supersymmetric quantum mechanics, and then summing over  $n$ , gives the factor of  $q^{1/4}\eta(q)^{-6} = \prod_{k=1}^{\infty}(1-q^k)^{-6}$  in (3.8). Including finally the sum over fluxes on  $\beta$ , we precisely reproduce the degeneracy (3.8)!

Let us now consider the more general case of D4-D2-D0 system. Again we shall assume  $(p^1, p^2, p^3) = (1, 1, 1)$ . The D2-brane charges are labelled by  $(q_1, q_2, q_3)$ . The bound state is described by the D4-brane with D2-brane dissolved in its world volume. We end up with the gauge flux

$$F = (m + 1/2)\beta + \sum_{i=1}^3 q_i \delta_i, \quad \delta_i \cdot \tilde{\alpha}_j = \delta_{ij}. \quad (4.5)$$

In above expression  $\delta_i$  is defined up to a shift of an integer multiple of  $\beta$ . Since we are summing over  $m$ , this ambiguity is irrelevant. We can choose  $\delta_i = \gamma - \tilde{\alpha}_i$ . The total induced D0-brane charge is then

$$\begin{aligned} \Delta q_0 &= - \int \frac{1}{2} F^2 - \frac{1}{4} \\ &= (m + 1/2)^2 + (m + 1/2) \sum q_i + \frac{1}{2} \sum q_i^2 - \frac{1}{4} \\ &= \left( m + \frac{1}{2} + \frac{1}{2} \sum q_i \right)^2 - \frac{1}{12} D^{AB} q_A q_B - \frac{1}{4}, \end{aligned} \quad (4.6)$$

where  $D^{AB}$  is the inverse matrix of  $D_{AB} \equiv D_{ABC} p^C$ ,

$$D^{AB} q_A q_B = 3(2q_1 q_2 + 2q_2 q_3 + 2q_3 q_1 - q_1^2 - q_2^2 - q_3^2). \quad (4.7)$$

Note that  $\frac{1}{3} D^{AB} q_A q_B = 0 \bmod 4$  if  $\sum q_i$  is even, and  $\frac{1}{3} D^{AB} q_A q_B = -1 \bmod 4$  if  $\sum q_i$  is odd. Therefore  $\Delta q_0$  is always an integer, as expected. The Cremmer-Julia invariant is in this case

$$\mathcal{J} = 4 \left( q_0 + \frac{1}{12} D^{AB} q_A q_B \right). \quad (4.8)$$

The counting of D0-brane states as before gives the generating function

$$\sum_{q_0} (-)^{\mathcal{J}} c(\mathcal{J}) q^{q_0} = - \prod_{k=1}^{\infty} (1 - q^k)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 + m - \frac{1}{12} D^{AB} q_A q_B} \quad (4.9)$$

in the case  $\sum q_i \in 2\mathbb{Z}$  and  $\mathcal{J} \equiv 0 \pmod{4}$ , and

$$\sum_{q_0} (-)^{\mathcal{J}} c(\mathcal{J}) q^{q_0} = - \prod_{k=1}^{\infty} (1 - q^k)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2 - \frac{1}{12} D^{AB} q_A q_B - \frac{1}{4}} \quad (4.10)$$

in the case  $\sum q_i \in 2\mathbb{Z} + 1$  and  $\mathcal{J} \equiv -1 \pmod{4}$ . These are precisely the degeneracies (3.10) we derived from 5D earlier!

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